



TITLE:

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CITATION:

NISHITANI, YASUAKI ...[et al]. The Firing Squad Synchronization Problem for Graphs (計算機構の数学的研究). 数理解析研究所講究録 1977, 296: 207-237

ISSUE DATE:

1977-05

URL:

<http://hdl.handle.net/2433/106221>

RIGHT:

# The Firing Squad Synchronization Problem for Graphs

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Abstract This paper deals with the Firing Squad Synchronization problem for some classes of digraph structures and graph structures. The first part of this paper gives solutions for the classes of circuit structures, quasi-circuit structures, and some other extended digraph structures. The second part gives a solution for the class of connected graph structures, whose synchronization time for a graph structure with radius  $r$  is  $3r+1$  or  $3r$  time units.

## 1. Introduction

The problem of synchronizing a finite (but arbitrarily long) one-dimensional array of finite automata, known as the firing squad synchronization problem, was proposed by Myhill and Moore [8]. Consider a one-dimensional array of identical finite automata. The state of each automaton at time  $t+1$

depends on its own state and those of its two neighbours at time  $t$ . The problem consists of defining the structure of automata so that one end automaton of the array, called the general, can cause all automata to enter a particular state, called the firing state, all at once.

It can easily be shown that the minimal time required to synchronize an  $n$ -element array is  $2n-2$  time units. The first minimal-time solution was obtained by Goto [2]. Waksman [13] has produced a minimal-time solution with 16 states and Balzer [1] has reduced the complexity to 8 states. The problem was generalized in many different ways by Moore and Langdon [9], Herman [3, 4], Rosenstiehl [11,12], Kobayashi [5, 6, 7], and Romani [10].

This paper deals with the firing squad synchronization problem for  $d$ -digraph structures and  $d$ -graph structures. Informally, a  $d$ -digraph structure ( $d$ -graph structure) is a network of identical finite automata in which an automaton is placed at each vertex of a digraph (graph) and the automata are connected along every arcs (edges) of the digraph (graph).

We present solutions of the problem for some subclasses of  $d$ -digraph structures in section 3 and for the class  $\Pi^d$  of connected  $d$ -graph structures in sections 4 and 5.

Rosenstiehl and Romani studied the problem of synchronizing a network of finite automata however connected. Rosenstiehl's solution obtains a synchronization time of  $2n$ , where  $n$  is the number of automata in the network, and Romani's solution

obtains a synchronization time shorter than or equal to that of Rosenstiehl's. The class of networks studied by Rosenstiehl and Romani is the same class as  $\Pi^d$  in our formulation. Our solution obtains a synchronization time of  $3r$  or  $3r+1$ , where  $r$  is the longest distance between the general and any other element in the network.

## 2. Preliminaries

In this section, we give definitions and notations used in this paper.

A digraph (or directed graph) is a pair  $(X, U)$ , where  $X$  is a set of elements called vertices and  $U$  is a set of ordered pairs of distinct vertices called arcs.

A graph (or undirected graph)  $G$  is a pair  $(X, E)$  where  $X$  is a set of vertices and  $E$  is a set of unordered pairs of distinct vertices called edges. A graph  $G$  is also regarded as a symmetric digraph  $G^*$  that has two oppositely directed arcs corresponding to each edge of  $G$ . In this paper we adopt this viewpoint. The order of a digraph (graph)  $G$ , denoted by  $|G|$ , is the number of vertices in  $G$ .

The distance from a vertex  $x$  to a vertex  $y$  in  $G$ , denoted by  $\text{dist}_G(x, y)$ , is the shortest length among the paths from  $x$  to  $y$ . Note that generally  $\text{dist}_G(x, y) \neq \text{dist}_G(y, x)$  in a digraph  $G$ .

A d-finite automaton  $M^d$  is a 6-tuple  $(S, s_e, s_q, s_g, s_f, \lambda)$ , where (1)  $S$  is a finite set of states, (2)  $s_e$  is an element not in  $S$  (the external signal), (3)  $s_g, s_q$ , and  $s_f$  are particular

distinct elements in  $S$  (the quiescent state, the general state, and the firing state respectively), and (4)  $\lambda$  is a transition function from  $S \times (S \cup \{s_e\})^d$  into  $S$  such that  $\lambda(s_q, s_1, \dots, s_d) = s_q$  if each of  $s_1, \dots, s_d$  is either  $s_q$  or  $s_e$ . Informally,  $M^d$  is an automaton with  $d$  input terminal. A  $d$ -tuple  $(s_1, \dots, s_d)$  in the set  $(S \cup \{s_e\})^d$  is called an input letter.

A  $d$ -digraph structure is a 3-tuple  $(G, x_g, d)$ , where  $d$  is a positive integer,  $G$  is a digraph such that  $d_G^- \leq d$  where  $d_G^-$  is the in-degree of  $G$ , and  $x_g$  is a particular vertex of  $G$  called the general. On a  $d$ -digraph structure, a  $d$ -finite automaton  $M^d$  is placed at each vertex of  $G$ . A vertex  $x$  installed with a  $d$ -finite automaton is called a cell  $x$ . Cells are connected along arcs in  $G$ . Let  $x$  be a vertex with  $d_G^-(x)$  into-arcs. Among  $d$  input terminals of a cell  $x$ ,  $d_G^-(x)$  of them are connected with the output terminals of the predecessor cells of  $x$  and the remaining  $d - d_G^-(x)$  ( $\geq 0$ ) input terminals are connected to the external world.

In order to describe clearly how the input terminals of  $x$  are connected to the predecessors of  $x$  or the external world, each input terminal is labeled a distinct integer  $i$  ( $1 \leq i \leq d$ ). If the input terminal of  $x$ , labeled  $i$ , is connected to a predecessor  $y$  of  $x$ ,  $y$  is called the  $i$ th predecessor cell of  $x$ . If the input terminal labeled  $i$  is connected to the external world, we say that the  $i$ th predecessor does not exist. The  $i$ th component of an input letter  $(s_1, \dots, s_d)$  of a cell  $x$  is the state of the  $i$ th predecessor cell of  $x$  (if it exists) or the external signal  $s_e$  (if not).

A d-digraph structure  $(G, x_g, d)$  is called a connected d-digraph structure if there is at least one path from  $x_g$  to  $y$  for any vertex  $y$  in  $G$ . In the followings, we shall deal with connected digraph structures, so we call them simply digraph structures.

Suppose that a d-digraph structure  $(G, x_g, d)$  is given and a d-finite automaton  $M^d$  is placed on each vertex of  $G$ . Then the state of a cell  $x$  at time  $t$ , denoted by  $s(x, t, G, x_g, M^d)$ , is defined by the following rules.

At  $t = 0$ , only the general cell  $x_g$  is in the general state  $s_g$  and all other cells are in the quiescent state  $s_q$ . That is,  $s(x, 0, G, x_g, M^d)$  is  $s_g$  if  $x = x_g$  and is  $s_q$  otherwise. Let

$$s_i = s(x_i, t, G, x_g, M^d) \quad \text{if the } i \text{ th predecessor } x_i \text{ of } x \text{ exists}$$

$$= s_e \quad \text{if it does not exist.}$$

Then the state of  $x$  at time  $t + 1$  is determined as

$$s(x, t + 1, G, x_g, M^d) = \lambda(s(x, t, G, x_g, M^d), s_1, \dots, s_d).$$

If the state of  $x$  at time  $t$  is  $s_f$ , we say that  $x$  fires at time  $t$ . The problem is to specify a automaton  $M^d$  which makes all cells in  $(G, x_g, d)$  to fire at once. A d-finite automaton  $M^d$  is called a solution of the firing squad synchronization problem for a subclass  $\Theta^d$  of d-digraph structures (simply a solution for  $\Theta^d$ ) if, for each d-digraph structures  $(G, x_g, d)$  in  $\Theta^d$ , there exists a time  $t(G, x_g, d, M^d)$  such that all cells in  $G$  fire at time  $t(G, x_g, d, M^d)$  and do not fire prior to time  $t(G, x_g, d, M^d)$ . The time  $t(G, x_g, d, M^d)$  is called the

synchronization time of  $M^d$  for firing a d-digraph structures  $(G, x_g, d) \in \Theta^d$ , (simply the synchronization time of  $M^d$  for  $(G, x_g, d)$ ).

Next, we define a d-graph structure  $(G, x_g, d)$ . Let  $G$  be a graph with  $d_G \leq d$ , and let  $x_g$  be a particular vertex of  $G$ . For  $G$ , we define the symmetric digraph  $G^*$  that has two oppositely directed arcs corresponding to each edge in  $G$ . Thus  $d_G = d_{G^*}^- = d_{G^*}^+$ . Then a d-graph structure  $(G, x_g, d)$  is defined to be the d-digraph structure  $(G^*, x_g, d)$ .

A d-graph structure  $(G, x_g, d)$  is called a connected d-graph structure if there is at least one path from  $x$  to  $y$  for any pair of distinct vertices. The class of connected d-graph structures and the corresponding d-digraph structures are denoted as  $\Pi^d$  and  $\Pi^{d*}$  respectively. In the followings, connected graph structures are called simply graph structures.

A d-finite automaton  $M^d$  is called a solution of the firing squad synchronization problem for  $\Pi^d$  if  $M^d$  is a solution for  $\Pi^{d*}$ . The synchronization time  $t(G, x_g, d, M^d)$  of  $M^d$  for a d-graph structure  $(G, x_g, d)$  is defined by the synchronization time  $t(G^*, x_g, d, M^d)$  for the corresponding d-digraph structure  $(G^*, x_g, d)$ . For a d-graph structures  $(G, x_g, d)$ , let  $t_{min}(G, x_g, d)$  be the minimum value of  $t(G, x_g, d, M^d)$  over all solutions  $M^d$  for  $\Pi^d$ . Given  $(G, x_g, d)$ , let  $L(G, x_g, d)$  be  $\max_{x,y} \{dist_G(x_g, x) + dist_G(x, y)\}$ . Kobayashi gave the following result about  $t_{min}(G, x_g, d)$  [5]

Theorem 2.1. For  $(G, x_g, d) \in \Pi^d$  with  $|G| = 1$ ,

$$t_{min}(G, x_g, d) = 1.$$

For  $(G, x_g, d) \in \Pi^d$  with  $|G| \geq 2$ ,

$$t_{\min}(G, x_g, d) \geq L(G, x_g, d).$$

Especially, if there are three cells  $x, x'$ , and  $y$  such that  $x$  and  $x'$  are adjacent,  $\text{dist}_G(x_g, x) = \text{dist}_G(x_g, x')$ , and  $\text{dist}_G(x_g, x) + \text{dist}_G(x, y) = \text{dist}_G(x_g, x') + \text{dist}_G(x', y) = L(G, x_g, d)$ , then

$$t_{\min}(G, x_g, d) \geq L(G, x_g, d) + 1.$$

Intuitively,  $L(G, x_g, d)$  is the time required for  $x$  to receive a signal from  $x_g$  and leave the quiescent state, and then for  $y$  to receive a signal from  $x$  for any vertices  $x$  and  $y$  in  $G$ .

### 3. Solutions for certain subclasses of d-digraph structures.

3.1 In this section, we give solutions for certain subclasses of d-digraph structures. A digraph  $C_n = (X_n, U_n)$  is called a circuit if  $X_n = \{x_0, \dots, x_{n-1}\}$ ,  $U_n = \{u_0, \dots, u_{n-1}\}$ ,  $u_i = (x_{i-1}, x_i)$  for each  $i$  ( $0 < i < n$ ), and  $u_0 = (x_{n-1}, x_0)$ . A circuit structure  $(C_n, x_0, 1)$  is a 1-digraph structure in which  $C_n$  is a circuit. Let  $\Theta_c$  be the class of circuit structures.

A solution for  $\Theta_c$  was given by Kobayashi. Its synchronization time for  $(C_n, x_0, 1)$  is  $2n-1$  time units. It is easily shown that the minimum time required to synchronize  $(C_n, x_0, 1)$  is  $2n-1$  time units. So the solution given by Kobayashi is a minimum time solution. The authors have obtained independently a similar solution. Here we give our solution  $M_c = (s_c, s_e, s_q, s_g, s_f, \lambda_c)$  which is called the circuit solution.

The evolution of the solution  $M_c$  is depicted in Fig. 1. The horizontal axis represents the circuit of cells in  $C_n$  and



the vertical axis represents time. The  $(z, t)$  entry represents the state of the  $z$  th cell at time  $t$ .

Let us divide cells in  $C_n$  into two equal parts. We shall represent a binary number  $n = a_0 + a_1 2 + \dots + a_m 2^m$  ( $a_i = 0$  or  $1$ ) as  $n = \langle a_1, \dots, a_m \rangle$ . In dividing  $n$  cells into two equal parts, each part is considered to contain  $\langle a_1, \dots, a_m \rangle$  cells. We divide the two halves into two parts each so that the size of each subdivision is  $\langle a_2, \dots, a_m \rangle$ . In similar fashion, the size of the  $k$  th subdivision is  $\langle a_k, \dots, a_m \rangle$ .

We use four signals  $P_{00}$ ,  $P_{11}$ ,  $P_{20}$ , and  $P_{21}$  for marking the boundaries between subdivisions and also for generating the following series of signals which propagate along the circuit.  $P_{00}$  and  $P_{11}$  are called general signals, and  $P_{20}$  and  $P_{21}$  are called subgeneral signals.

A general signal  $P_{00}$  generates following series of signals: a  $P$ -series consisting of  $P_0$  and  $P_1$  signals which does not propagate,

$BC$ -series consisting of  $B_0, B_1, B_2, B_3, C_0, C_1, C_2$ , and  $C_3$  signals which propagate with velocities  $v = 1/3, 3/7, \dots, (2^i - 1)/(2^{i+1} - 1), \dots$  (cells/time unit), (a  $BC$ -series which propagates with  $v = (2^i + 1)/(2^{i+1} - 1)$  is called a  $(BC)_i$ -series), an  $A_0$ -series consisting of  $A_{00}$  and  $A_{01}$  signals which propagates with  $v = 1$ , and

$RS$ -series consisting of  $R_1, R_2, S_0, S_1$ , and  $S_2$  signals which propagate with  $v = 2/3, 4/7, \dots, 2^i/(2^{i+1} - 1), \dots$  ( $i = 1, 2, \dots$ ), (an  $RS$ -series which propagates with  $v = 2^i/(2^{i+1} - 1)$  is called an  $(RS)_i$ -series).

A general signal  $P_{11}$  generates following series of signals:  
a  $P$ -series,

$BC$ -series which propagate with the same velocities as those of the above  $BC$ -series but are delayed one time unit,  
an  $A_1$ -series consisting of  $A_{10}$  and  $A_{11}$  signals which propagates with  $v = 1$ , and  $RS$ -series.

A subgeneral signal  $P_{2l}$  ( $l = 0$  or  $1$ ) at  $(z, t)$  generates  $P_{2l}$  signal at  $(z+1, t+1)$  and a  $P_2$ -series consisting of  $P_2$  signals which does not propagate.

A  $(BC)_i$ -series is obtained if we delay a series, which propagates with  $v = 1/2$ , one unit time on every  $2^{i-1} - 1$  cells. It is shown that a  $(BC)_{i+1}$ -series is produced by a  $(BC)_i$ -series inductively.

A  $(RS)_i$ -series is obtained if we advance a series, which propagates with  $v = 1/2$ , one unit time on every  $2^i$  cells.

It is shown that a  $(RS)_{i+1}$  series is produced by a  $(RS)_i$ -series inductively.

We shall show how general and subgeneral signals are generated on boundaries of subdivision. General and subgeneral signals are generated according to the following rules.

- (1) When an  $A_0$ -series meets  $C_2$  of a  $BC$ -series,  $P_{00}$  is generated.
- (2) When an  $A_0$ -series meets  $B_2$  of a  $BC$ -series,  $P_{11}$  is generated.
- (3) When an  $A_1$ -series meets  $C_3$  or  $C_0$  of a  $BC$ -series,  $P_{20}$  is generated.
- (4) When an  $A_1$ -series meets  $B_3$  or  $B_0$  of a  $BC$ -series,  $P_{21}$

is generated.

- (5) When a  $P$ -series meets  $S_2$  or  $S_0$  of an  $RS$ -series,  $P_{00}$  is generated.
- (6) When a  $P$ -series meets  $R_2$  of an  $RS$ -series,  $P_{11}$  is generated.
- (7) When a  $P_2$ -series meets  $S_2$  or  $S_0$  of an  $RS$ -series,  $P_{20}$  is generated.
- (8) When a  $P_2$ -series meets  $R_2$  of an  $RS$ -series,  $P_{21}$  is generated.
- (9) When a  $P$ -series meets  $A_0^*$  of an  $A_0$ -series,  $P_{**}$  is generated ( $*$  = 0 or 1).

Four cases are to be considered.

Case 1. let  $n$  be an integer represented by  $\langle a_0, \dots, a_m \rangle$ , and suppose that  $P_{00}$  is generated at  $(0, 0)$  and  $P_{a_0 a_0}$  is generated at  $(n, n)$ . It is shown that  $P_{a_i a_i}$  is generated at  $(n, 2n - \langle a_i, \dots, a_m \rangle)$  ( $i = 1, 2, \dots$ ).

Case 2. Suppose that  $P_{00}$  is generated at  $(0, 0)$  and  $P_{a_0 a_0}$  is generated at  $(0, n)$ . It is shown that  $P_{a_i a_i}$  is generated at  $(n - \langle a_i, \dots, a_m \rangle, 2n - \langle a_i, \dots, a_m \rangle)$ , and if  $a_{i-1} = 1$ ,  $P_{2a_i}$  is generated at  $(n - \langle a_i, \dots, a_m \rangle - 1, 2n - \langle a_i, \dots, a_m \rangle - 1)$ .

Case 3. Suppose that  $P_{11}$  is generated at  $(0, 0)$ ,  $P_{2a_0}$  is generated at  $(n, n)$ , and  $P_{a_0 a_0}$  is generated at  $(n+1, n+1)$ . It is shown that  $P_{2a_i}$  is generated at  $(n, \langle a_i, \dots, a_m \rangle)$  by the similar way as in case 1 and  $P_{a_i a_i}$  is generated at  $(n+1, 2n - \langle a_i, \dots, a_m \rangle + 1)$ .

Case 4. Suppose that  $P_{11}$  is generated at  $(0, 0)$  and

$P_{a_0 a_0}$  is generated at  $(0, n+1)$ . By the similar way as in case 2 and by considering that the  $BC$ -series generated by  $P_{11}$  propagate with one time unit delay, it is shown that  $P_{a_i a_i}$  is generated at  $(n - \langle a_i, \dots, a_m \rangle, 2n - \langle a_i, \dots, a_m \rangle + 1)$  and if  $a_{i-1} = 1$ ,  $P_{2a_i}$  is generated at  $(n - \langle a_i, \dots, a_m \rangle - 1, 2n - \langle a_i, \dots, a_m \rangle)$ .

From the above consideration, we conclude that the general or subgeneral signals are generated synchronously at boundaries of subdivisions. Then, it is seen that all cells fire at time  $(2n-1)$  for  $(c_n, x_0, 1)$  and thus  $M_c$  is a solution for  $\theta_c$ .

Theorem 3.1.  $M_c = (s_c, s_e, s_g, s_q, s_f, \lambda_c)$  is a solution for the class  $\theta_c$  of circuit structures and its synchronization time for  $(c_n, x_0, 1)$  is  $2n-1$  time units. The number of states of  $M_c$  is 38.

The evolution of the circuit solution  $M_c$  for  $(c_{13}, x_0, 1)$  is given in Fig. 2, where  $s_g$ ,  $s_f$ , and  $s_q$  are denoted by  $P_{00}$ ,  $F$ , and blank respectively.

3.2. We consider a digraph  $C_n^1 = (X_n^1, U_n^1)$  which is called a quasi-circuit. A quasi-circuit  $C_n^1 = (X_n^1, U_n^1)$  is defined as follows.

$$(1) \quad X_n^1 = \{x_{ij} \mid 0 \leq i \leq n-1, 0 \leq j \leq h_i, h_0 = 0\}.$$

$$(2) \quad U_n^1 = \{(x_{i-1, 0}, x_{i0}) \mid 0 \leq i \leq n-1, x_{-1, 0} = x_{n-1, 0}\}$$

$$\bigcup_{0 \leq i \leq n-1} \bigcup_{0 \leq j \leq h_j} U_{ij},$$

where  $U_{ij}$  is the set of arcs of the form  $(x_{i-1, k}, x_{ij})$  for some  $k$ .

(3) There exists at least one path from  $x_{00}$  to  $x_{ij}$  for all vertices  $x_{ij}$ .

From the definition, it is shown that

- (1)  $\text{dist}_{C_n^1}(x_{00}, x_{ij}) = i < n$ ,
- (2) there exists at least one circuit in  $C_n^1$ , and
- (3) all circuits of  $C_n^1$  pass through  $x_{00}$  and their length are  $n$ .

A d-quasi-circuit structure is a d-digraph structure  $(C_n^1, x_{00}, d)$  where  $C_n^1$  is a quasi-circuit. Let  $\Theta_{C_1}^d$  be the class of d-quasi-circuit structures. A solution for  $\Theta_{C_1}^d$  can be obtained by slightly modifying the circuit solution  $M_c$ . Let  $M_{C_1}^d = (s_c, s_e, s_g, s_q, s_f, \lambda_{C_1}^d)$  be a d-finite automaton whose input letters are d-tuples. The state transition function  $\lambda_{C_1}^d$  is defined only for such input letters that all components other than  $s_e$  are identical signals. For these defined inputs,  $M_{C_1}^d$  behaves as  $M_c$  does. In more detail, let an input letter of  $M_{C_1}^d$  whose every component is either  $s \in S_c$  or  $s_e$  be expressed as  $s^d$ . We define  $\lambda_{C_1}^d(s', s^d) = \lambda_c(s', s)$  for all  $s', s \in S_c$  where  $\lambda_c$  is the state transition function of  $M_c$ .

Theorem 3.2.  $M_{C_1}^d$  is a solution for  $\Theta_{C_1}^d$  and its synchronization time for  $(C_n^1, x_{00}, d) \in \Theta_{C_1}^d$  is  $2n-1$  time units.

Proof. It is easily proved by the induction on  $t$  that at any time  $t$  and for each  $i$  ( $0 \leq i \leq n-1$ ), the state of the automaton at  $x_{ij}$  is independent of  $j$ , that is,  $s(x_{ij}, t, C_n^1, x_{00}, M_{C_1}^d)$  is identical for all  $j$  ( $0 \leq j \leq h_i$ ).

Since there is at least one circuit in  $(C_n^1, x_{00}, d)$ , all cells on the circuit fire at time  $2n-1$ . Hence all cells in

$(C_n^1, x_{00}, d)$  fire at time  $2n-1$ .  $M_{c_1}^d$  is called the quasi-circuit solution.

3.3 We consider a digraph  $C_n^2 = (X_n^2, U_n^2)$  which is defined as follows.

- (1) There is at least one circuit in  $C_n^2$  and all circuits in  $C_n^2$  pass through a designated vertex  $x_{00}$ .
- (2) The maximum length of circuits in  $C_n^2$  is  $n$ .
- (3) For each vertex  $x$ , there is at least one path from  $x_{00}$  to  $x$  and the maximum length of paths, in which no vertex is encountered more than once, is less than  $n$ .

In other words,  $C_n^2$  is obtained by adding arcs of the form  $(x_{ij}, x_{i'j'})$  with  $i < i'-1$  to a quasi-circuit  $C_n^1$ .

Let  $\Theta_{c_2}^d$  be the class of d-digraph structures  $(C_n^2, x_{00}, d)$  and  $M_{c_2}^d$  be the d-finite automaton which is obtained by modifying the quasi-circuit solution  $M_{c_1}^d$  as explained below.  $M_{c_2}^d$  consists of  $M_{c_1}^d$  and the processor for the input signal. The processor finds which predecessor cells move to non-quiet state in  $M_{c_1}^d$  lastly. Since then, the processor regards the signals from the predecessors other than the lastly activated ones as external signals. In other words,  $M_{c_2}^d$  disregards input signals received through arcs  $(x_{ij}, x_{i'j'})$  with  $i < i'-1$  stated for defining  $C_n^2$  from  $C_n^1$ . Then Theorem 3.3 is easily proved

Theorem 3.3  $M_{c_2}^d$  is a solution for  $\Theta_{c_2}^d$  and its synchronization time for  $(C_n^2, x_{00}, d) \in \Theta_{c_2}^d$  is  $2n-1$  time units.

3.4 We consider a digraph  $C_n^3 = (X_n^3, U_n^3)$  which is defined as follows.

- (1) There exists at least one circuit which pass through a

designated vertex  $x_{00}$ .

(2) The minimum length of circuits passing through  $x_{00}$  is  $n$ .

(3) For each  $x$ , there exists at least one path from  $x_{00}$  to  $x$  and  $\text{dist}_{C_n^3}(x_{00}, x) < n$ .

Let  $\Theta_{C_n^3}^d$  be the class of d-digraph structures  $(C_n^3, x_{00}, d)$  and  $M_{C_n^3}^d$  be the d-finite automaton which consists of the quasi-circuit solution  $M_{C_n^3}^d$  and the processor for the input signal.

The processor finds which predecessor cells move to non-quietcent states first. Since then, the processor regards the signals from the predecessors other than the first activated ones as external signals. Then Theorem 3.4 is easily proved.

Theorem 3.4  $M_{C_n^3}^d$  is a solution for  $\Theta_{C_n^3}^d$  and its synchronization time for  $(C_n^3, x_{00}, d)$  is  $2n-1$  time units.

#### 4. Two preliminary solutions for d-graph structures.

4.1. In this section, we shall consider the class  $\Pi^d$  of d-graph structures, and give two preliminary solutions  $M_{4r}^d$  and  $M_{3r+1}^d$  for  $\Pi^d$ . Let  $(G_r, x_g, d)$  be a d-graph structure with the radius  $r$ . Here, the radius  $r$  of a graph structure  $(G, x_g, d)$  is defined by  $r = \max_{x \in G} \text{dist}_G(x_g, x)$ . It will be shown that the synchronization time of  $M_{3r+1}^d$  for  $(G_r, x_g, d)$  is  $3r+1$  time units. We call  $M_{3r+1}^d$  a  $3r+1$  solution.

Before explaining the essential idea for constructing  $M_{3r+1}^d$ , we shall give a preliminary solution  $M_{4r}^d$  whose synchronization time for  $(G_r, x_g, d)$  is  $4r$  time units. We call  $M_{4r}^d$  a  $4r$  solution. The principal idea is to construct the automaton

which reduces a given d-graph structure to a d-quasi-circuit structures and then simulate the quasi-circuit solution  $M_{c1}^d$ .

In  $(G, x_g, d) \in \Pi^d$ , if a cell  $x$  has no adjacent cell  $y$  such that  $\text{dist}_G(x_g, y) > \text{dist}_G(x_g, x)$ , then  $x$  is called a terminal cell. For each cell  $x$ , there is at least one path  $\mu = [x_0, x_1, \dots, x_l]$  such that  $x_0 = x$ ,  $x_l$  is a terminal cell, and for all  $j$  ( $0 \leq j < l$ ),  $\text{dist}_G(x_g, x_j) < \text{dist}_G(x_g, x_{j+1})$ . When  $x_1$  is the  $i$ th adjacent cell of  $x = x_0$ , the path  $\mu$  is called the  $i$ th path of  $x$ . The maximum length of the  $i$ th paths of  $x$  is denoted by  $m(x, i)$ . Note that if  $x$  is a terminal cell,  $m(x, i)$  is 0 for all  $i$ .

A d-graph structure  $(G_r, x_g, d) \in \Pi^d$  is reduced to a d-quasi-circuit structure  $(C_{2r}^1, x_g, d) \in \Theta_{c1}^d$  as follows.

(See Fig. 3.)

First, we remove every edge  $e = [x, y]$  in  $G_r$  such that  $\text{dist}_{G_r}(x_g, x) = \text{dist}_{G_r}(x_g, y)$  and obtain an d-graph  $G'_r$ . Then we divide each cell  $x$  other than the general cell  $x_g$  and terminal cells into two subcells  $x^1$  and  $x^2$  called the first subcell and the second subcell respectively and replace each edge  $e = [x, y]$  in  $G'_r$ , for which  $\text{dist}_{G'_r}(x_g, x) < \text{dist}_{G'_r}(x_g, y)$ , with two arcs  $u^1 = (x^1, y^1)$  and  $u^2 = (y^2, x^2)$ . (For the general cell (terminal cells),  $x^1(y^1) = x^2(y^2) = x(y)$ .) Finally, for each  $x$  in  $G'_r$  and  $i$  ( $1 \leq i \leq d$ ), if there exists  $j$  such that  $m(x, i) < m(x, j)$ , then we remove  $(x_i^2, x)$  where  $x_i^2$  is the second subcell of the  $i$ th adjacent cell  $x_i$  of  $x$ . Then we obtain a quasi-circuit  $C_{2r}^1$ .

The solution  $M_{4r}^d = (S_{4r}, s_e, s_g, s_q, s_f, \lambda_{4r})$  first



simulates the above reducing process. Its state set  $S_{4r}$  is given by  $S_1 \times S_2 \times S_3 \times S_4 \cup \{s_f\}$ .  $s_f$  is the firing state of  $M_{4r}^d$ .  $S_1$  and  $S_2$  are used to simulate the reducing process.  $S_3$  and  $S_4$  are used to simulate  $M_{ci}^d$  on the d-quasi-circuit structure reduced from a given d-graph structure.

We put

$$S_1 = \{G_0, G_1, G_2, H_0, H_1, I, J, Q_0\},$$

$$S_2 = \{0, 1, 2, 3\}^d,$$

$$S_3 = S_4 = S_c - \{F\},$$

where  $S_c$  is the state set of the quasi-circuit solution  $M_{ci}^d$  and  $F$  is its firing state.

The general cell starts from  $G_0$  and reaches to  $G_2$  via  $G_1$  and each terminal cell starts from  $Q_0$  and reaches to  $I$  via  $H_0$ . Each non-terminal cell starts from  $Q_0$  and reaches to  $J$  through  $H_0$  and  $H_1$ .

Each element of  $S_2$  is expressed as  $(m_1, \dots, m_i, \dots, m_d)$  where  $m_i \in \{0, 1, 2, 3\} (1 \leq i \leq d)$ . Let  $x$  be a cell and  $x_i$  be the  $i$ th adjacent cell of  $x$ . Let the  $S_2$  component of  $x$  is  $(m_1, \dots, m_d)$ . The value of  $m_i$  has the following meaning.  $m_i = 1$  means that  $\text{dist}_G(x_g, x) < \text{dist}_G(x_g, x_i)$  and thus the arcs  $(x_i^1, x^1)$  exists in  $C_{2r}^1$ .  $m_i = 2$  means that  $\text{dist}_G(x_g, x) < \text{dist}_G(x_g, x_i)$  and thus the arcs  $(x^1, x_i^1)$  and  $(x_i^2, x^2)$  exist in  $C_{2r}^1$ .  $m_i = 3$  means either that  $\text{dist}_G(x_g, x) = \text{dist}_G(x_g, x_i)$  or  $x_i$  does not exist and thus the edge  $(x, x_i)$  is removed from  $G_r$ , or that there exists  $i$  such that  $m(x, j) > m(x, i)$  and thus the arc  $(x_i^2, x^2)$  is removed from  $G_1$ .  $m_i = 0$  means that the connection between  $x$  and  $x_i$  is not yet determined.

Initially, the general cell is in the state  $(G_0, (0, \dots, 0))$  and other cells are in  $(Q_0, (0, \dots, 0))$ .

In simulating the behaviors of  $M_{c1}^d, M_{4r}^d$  makes each subcell to receive input signals only from its predecessor subcells in  $C_{2r}^1$  and ignore those from other subcells.

It is shown that  $M_{4r}^d$  can simulate the reduction from  $G_r$  to  $C_{2r}^1$  and the solution  $M_{c1}^d$  on the d-quasi-circuit structure  $(C_{2r}^1, x_g, d)$ . Fig. 3 illustrates the solution  $M_{4r}^d$  on a  $(G_3, x_g, 3)$ .

Let  $dist_G(x_g, x)$  and  $\max_i m(x, i)$  be denoted by  $l_x^1$  and  $l_x^2$ . It is seen that the arcs incident into  $x^1$  are established at  $t = l_x^1$  and arcs incident into  $x^2$  are established at  $t = l_x^1 + 2l_x^2 + 1$ . Hence, the general cell can start to simulate  $M_{c1}^d$  at  $t = 1$ .

We define that each cell moves to  $s_f$  when its two subcells move to  $F$ . Then the synchronization time of  $M_{4r}^d$  for  $(G_r, x_g, d)$  is  $1 + (2(2r) - 1) = 4r$  time units.

Theorem 4.1  $M_{4r}^d$  is a solution for  $\Pi^d$  and its synchronization time for  $(G_r, x_g, d)$  is  $4r$  time units if  $|G_r| \geq 2$  and 1 if  $|G| = 1$ .

Next, we shall describe the  $3r + 1$  solution  $M_{3r+1}^d$ .  $M_{3r+1}^d$  simulates the reduction from  $(G_r, x_g, d)$  to  $(C_{2r}^1, x_g, d)$  as  $M_{4r}^d$  does. Kobayashi pointed out the following two facts about  $(C_{2r}^1, x_g, d)$  and suggested that the synchronization time for  $(G_r, x_g, d)$  is improved to about  $3r$  time units.

(1) Synchronization of  $(G_r, x_g, d)$  is achieved by synchronizing the subdigraph structure of  $(C_{2r}^1, x_g, d)$  consisting

only of the first cells.

(2) A terminal cell  $x_M$  farthest from  $x_g$  divides a circuit in  $(C_{2^n}^1, x_g, d)$  into two halves.  $x_M$  is called the center of  $C_{2^n}^1$ . We can find the center of  $C_{2^n}^1$  at time  $n$  in the reduction process.

Considering the two facts, we have the following modified problem. Let  $(C_{2^n}, x_0, x_n, 1)$  be a circuit structure in which the center  $x_n$  of  $C_{2^n}$  is designated. Let  $(x_0, x_1, \dots, x_n, \dots, x_{2^n-1})$  be the circuit. Find a solution of synchronizing all cells on the semicircuit  $(x_0, x_1, \dots, x_n)$  for the class of  $(C_{2^n}, x_0, x_n, 1)$ .

We shall give a 1-finite automaton  $M_h = (S_h, s_e, s_q, s_g, s_f, \lambda_h)$ , called the semicircuit solution whose synchronization time for  $(C_{2^n}, x_0, x_n, 1)$  is  $3n-1$  time units.  $M_h$  is essentially similar to the circuit solution  $M_c$ , and  $S_h$  includes  $S_c$ .

The evolution of the solution  $M_h$  on  $(C_{2^n}, x_0, x_n, 1)$  is depicted in Fig. 4, in which the evolution of  $M_c$  on  $(C_n, x_0, 1)$  is also shown for the reference. It is shown that the signals generated at  $(z, t)$  in  $(C_{2^n}, x_0, x_n, 1)$  is identical to those at  $(z, t-n)$  in  $(C_n, x_0, 1)$  for  $0 \leq z \leq n-1$  and  $2n + z \leq t$ .

Moreover, the center cell  $x_n$  fires at  $t = 3n-1$ . Hence all cells on the semicircuit  $(x_0, \dots, x_n)$  of  $C_{2^n}$  fire at time  $3n-1$  simultaneously. Fig. 5 gives the semicircuit solution for  $(C_{12}, x_0, x_6, 1)$ .

Next, we consider a d-quasi-circuit structure  $(C_{2^n}^1, x_{00}, X_n, d)$  in which  $X_n$  is the set  $\{x_{nj}\}$  of the center of circuits in  $C_{2^n}^1$  and all  $x_{nj}$ 's are designated. A solution for synchronizing

all cells in  $X_n$  and all cells  $x_{ij}$ 's, where  $0 \leq i \leq n-1$ ,  $0 \leq j \leq h_i$ , is given by an d-finite automaton  $M_h^d$  which is obtained by slightly modifying  $M_h$ .

$M_h^d$  is defined from  $M_h$  by the same way as the quasi circuit solution  $M_c^d$  is defined from the circuit solution  $M_c$ . That is, the state transition function of  $M_h^d$  is defined for such input letters that all components other than the external signal are identical signals, and for these input letters  $M_h^d$  behaves as  $M_h$  does.

By the similar arguments used for proving Theorem 3.2, it is easily shown that  $M_h^d$  is a solution for the above problem and its synchronization time for  $(C_{2n}^1, x_{00}, X_n, d)$  is  $3n-1$  time units.  $M_h^d$  is called the quasi-semicircuit solution.

Now, we shall give a  $3r+1$  solution  $M_{3r+1}^d$  for the class of d-graph structures  $(G_r, x_g, d)$  by using the concept of  $M_{4r}^d$  and  $M_h^d$ . The state set  $S_{3r+1}$  of  $M_{3r+1}^d$  is expressed as  $(S_1 \times S_2 \times \dots \times S_{30} \times S_{40}) \cup \{s_f\}$  where  $S_1$  and  $S_2$  is the same set as  $S_{4r}$  of  $M_{4r}^d$ ,  $S_{30} = S_{40}$  is the same set as  $S_h^d$  of  $M_h^d$ , and  $s_f$  is the firing state.

Given a d-graph structure  $(G_r, x_g, d)$ ,  $M_{3r+1}^d$  starts at  $t = 0$  to reduce  $(G_r, x_g, d)$  to a d-quasi-circuit structure  $(C_{2r}^1, x_g, d)$  as  $M_{4r}^d$  does and also starts at  $t = 1$  to simulate the quasi-semicircuit solution  $M_h^d$ . The synchronizing time of  $M_{3r+1}^d$  appears to be  $3r$  time units, but it is not.

Let  $x$  be a terminal cell for which  $\text{dist}_G(x_g, x) = r' < r$ . Since  $x$  moves to  $(H_0, *, *)$  at time  $r'$  and moves to  $(I, *, *, *)$  at time  $r' + 1$  for  $M_{4r}^d$ ,  $x$  moves to  $(I, *, P'_{00}, P'_{00})$  at time  $r' + 1$  for  $M_{3r+1}^d$ . Then the subcells of  $x$  move to the

firing state  $F$  of  $M_h^d$  at time  $3r' < 3r$ , while all other subcells move to  $F$  at time  $3r$ . Thus it fails to synchronize all the first subcells of  $G_r$ .

Since  $x$  has at least one adjacent cell *e.g.*  $i$ th cell with  $m_i = 1$  which does not move to  $F$  prior to time  $3r$ . ( $m_i$  is the  $i$ th component of  $(m_1, \dots, m_d) \in S_2$  of  $x$ .) We define the firing of  $M_{3r+1}^d$  as follows. Let  $x$  be any cell in  $(G_r, x_g, d)$ . When the first subcell of  $x$  is  $F$  and the first subcells of all adjacent cells of  $x$  are also in  $F$  at time  $t$ ,  $x$  moves to the firing state  $s_f$  of  $M_{3r+1}^d$  at time  $t + 1$ . It requires one more time unit.

Theorem 4.2 The d-finite automaton  $M_{3r+1}^d$  is a solution for  $\Pi^d$  and its synchronization time for  $(G_r, x_g, d) \in \Pi^d$  is  $3r+1$  time units.

##### 5. A $3r$ solution for d-graph structures.

In this section, we give a improved solution  $M_{3r}^d$  for  $\Pi^d$  whose synchronization time for  $(G_r, x_g, d) \in \Pi_s^d$  is  $3r$  time units, where  $\Pi_s^d$  is a subclass of  $\Pi^d$ .

We call a cell  $x$  in  $G_r$ , for which  $\text{dist}_{G_r}(x_g, x) = r$ , a radial cell. A cell  $x$  in  $G$ , for which there exists no cell  $y$  such that  $\text{dist}_G(x_g, y) \geq \text{dist}_G(x_g, x)$ , is called a solitary cell.

The reason why  $M_{3r+1}^d$  requires one more time unit for the synchronization is that the first subcells of non-radial terminal cells move to  $F \in S_h$  before time  $3r$ . We shall consider to overcome this difficulty without loss of a time unit.

Let  $x$  be a terminal cell for which  $\text{dist}_{G_r}(x_g, x) = r'$ . In  $M_{3r}^d$ , the first and second subcells of  $x$  move respectively to  $A$  and  $P_{00}' \in S_h$  at time  $r'+1$ . This is achieved by slightly modifying  $M_{3r+1}^d$ . In other words, the first subcell behaves as if  $x$  is non-radial and the second one does as if  $x$  is radial.

For a non-radial terminal cell  $y$ , the first subcell of  $y$  moves to  $F$  at time  $3r$  and the second one moves to  $F$  at time  $3r'$ . For a radial cell  $x$ , the first subcell of  $x$  does not move to  $F$  prior to time  $3r$  and the second one moves to  $F$  at time  $3r$ . For a non-terminal cell, the first subcell moves to  $F$  at  $t = 3r$ .

From the above consideration, if each cell  $x$  recognizes prior to time  $3r'$  ( $r' = \text{dist}_{G_n}^+(x_g, x)$ ) whether it is radial or not then all cells can fire once at time  $3r$ .

Usually, a terminal cell  $x$  requires  $r'$  time units ( $r' = \text{dist}_{G_r}(x_g, x)$ ) to recognize that it is terminal and hence requires  $3r'$  time units to recognize whether it is radial or not. But a solitary non-radial cell  $x$  requires  $r'-1$  time units to recognize that it is non-radial or there exists at least one non-solitary terminal cell  $y$  such that  $\text{dist}_{G_r}(x_g, y) = r'$ . If all radial cells are solitary, then each of them recognizes at time  $3r-1$  that it is radial and all other cells recognize prior to time  $3r-1$  that all radial cells are solitary. Thus for  $(G_r, x_g, d)$  in which every radial cell is solitary, we can obtain a solution whose synchronization time is  $3r$  time units.

Let  $\Pi_s^d$  be a subclass of  $\Pi^d$  consisting of  $d$ -graph structures in which all radial cells are solitary. We shall give a solution

$M_{3r}^d$  whose synchronization time for  $(G_r, x_g, d)$  of  $\Pi_s^d$  and  $\Pi^d - \Pi_s^d$  are respectively  $3r$  and  $3r+1$  time units.

The fundamental behaviors of  $M_{3r}^d$  is identical to that of  $M_{3r+1}^d \cdot M_{3r}^d$  simulates the reduction from a  $d$ -connected graph structure to a  $d$ -quasi circuit structure and then simulates the behaviors of the quasi-semicircuit solution. Beside these behaviors, when the general cell recognizes that all radial cells are solitary, it sends signals about this knowledge to all other cells.

The state set of  $M_{3r}^d$  is  $(S_{10} \times S_2 \times S_3 \times S_4) \cup \{s_f\}$ .  $S_{10} = S_1 \cup \{G_{20}, I_0, I_1, I_2, I_{20}, J_0, J_1, J_2, J_{20}\}$ .  $S_1$  and  $S_2$  are given in  $M_{4r}^d$  and  $S_3 = S_4 = s_h$  are given in  $M_h$ .  $s_f$  is the firing state of  $M_{3r}^d$ . The states in  $S_{10}$  play the following roles. The general cell moves to  $G_2$  when it recognizes all radial cells to be solitary, or moves to  $G_{20}$  when it finds at least one non-solitary radial cell. If a terminal cell  $x$  recognizes itself to be solitary, then  $x$  moves to  $I_2$  via  $I_0$  and  $I_1$ , else  $x$  moves to  $I$ . When  $x$  in  $I_2$  recognizes itself not to be radial or it finds at least one non-solitary terminal cell  $y$  such that  $dist_{G_r}(x_g, y) = dist_{G_r}(x_g, x)$ ,  $x$  moves to  $I$ . When a cell in  $I$  recognizes that all radial cells are solitary, it moves to  $I_{20}$ .  $J, J_0, J_1$  and  $J_2$  play the same roles for non-terminal cells as  $I, I_0, I_1$  and  $I_2$  do for terminal cells. When a cell in  $J_2$  recognizes that all radial cells are solitary, it moves to  $J_{20}$ . Thus, for any  $d$ -graph structure in  $\Pi_s^d$ , the general cell, radial cells, non-radial terminal cells, and the non-terminal cells move respectively to  $G_{20}, I_2, I_{20}$ , and  $J_{20}$  one time unit before their

firing.

Signals in  $S_1$  play the same roles for  $M_{3r}^d$  as it does for  $M_{4r}^d$ . Signals  $I_0$ ,  $I_1$ , and  $I_2$  are generated at solitary cells. Signals  $G_{20}$ ,  $J_{20}$ , and  $I_{20}$  are generated when all radial cells are found to be solitary and are used to transmit this knowledge. Initially, the general cell is in the state  $(G_0, (0, \dots, 0) Q, Q)$  and other cells are in  $(Q_0, (0, \dots, 0), Q, Q)$ .

In simulating the behaviors of the d-quasi-semicircuit solution  $M_h^d$ ,  $M_{3r}^d$  makes each subcell to receive input signals only from its predecessor subcells and ignore input signals from other subcells. This can be achieved by the method similar to those used in  $M_{4r}^d$ .

If a cell  $x$  in  $(G_r, x_g, d)$  with  $\text{dist}_G(x_g, x) = r'$  is solitary,  $x$  moves to  $I_0$  at time  $r'$  and sends a  $J_0$  series to the general cell  $x_g$ . Thus, if all radial cells are solitary, then  $x_g$  moves to  $G_{20}$  at time  $2r$  and sends  $J_{20}$  signals to all cells in  $(G_r, x_g, d)$ , else  $x_g$  moves to  $G_2$  at time  $2r+1$ . As a result, if all radial cells are solitary, they are in  $I_2$  and all other cells are in  $G_{20}$ ,  $J_{20}$ , or  $I_{20}$  at time  $3r-1$ . (For  $r = 1$ , radial cells are in  $I_1$ , instead of  $I_2$ .) On the otherhand, if not all of radial cells are solitary, then solitary radial cells are in  $I_2$  and all other cells are in  $G_2$ ,  $J$ , or  $I$  at time  $3r-1$ . (For  $r = 1$ , solitary radial cells move to  $I_1$ .)

Note that if all radial cells are solitary, then they can recognize themselves to be radial at time  $3r-1$ . Hence we define the firing of  $M_{3r}^d$  as follows.

- (1) If  $x$  is in  $G_{20}$ ,  $J_{20}$ , or  $I_{20}$  at time  $t$  and the first



subcell of  $x$  moves to  $F \in S_{30} = S_h$  at time  $t + 1$ , then  $x$  moves to  $s_f$  at time  $t + 1$ .

(2) If  $x$  is in  $I_2$  or  $I_1$ , all  $x_i$ 's with  $m_i = 1$  are in  $J_{20}$  or  $G_{20}$  at time  $t$ , and the second subcell of  $x$  moves to  $F$  at time  $t + 1$ , then  $x$  moves to  $s_f$  at time  $t + 1$ .

(3) If  $x$  is in  $G_2$  or  $J$  and the first subcell of  $x$  moves to  $F$  at time  $t$ , then  $x$  moves to  $s_f$  at time  $t + 1$ .

(4) If  $x$  is in  $I$  and the first subcells of  $x_i$ 's with  $m_i = 1$  move to  $F$  at time  $t$ , then  $x$  moves to  $s_f$  at time  $t + 1$ .

If all radial cells are solitary, then all cells in  $(G_r, x_g, 1)$  fire at time  $3r$  according to (1) and (2), else all cells fire at time  $3r+1$  according to (3) and (4). (See Figs. 6 and 7))

Theorem 4.1  $M_{3r}^d$  is a solution for  $\Pi^d$ . The synchronization time of  $M_{3r}^d$  is 1 time unit for  $(G, x_g, d) \in \mathbb{H}^d$  with  $|G| = 1$ ,  $3r$  time units for  $(G_r, x_g, d) \in \Pi_s^d$  with  $|G| \geq 2$ , and  $3r+1$  time units for  $(G_r, x_g, d) \in \Pi^d - \Pi_s^d$  with  $|G| \geq 2$ .

Finally, we shall show that  $M_{3r}^d$  give the minimum synchronization time for some subclasses of  $\Pi_s^d$ . Let  $(G_r, x_g, d)$  be a member of  $\Pi_s^d$  which has two radial cells such that  $\text{dist}_G(x_1, x_2) = 2r$ . We denote the set of such  $d$ -connected graph structures by  $\Pi_{3r}^d$ . Theorem 1.1 shows that

$$t_{\min}(G_r, x_g, d) \geq L(G_r, x_g, d) = 3r$$

for any  $(G_r, x_g, d)$  in  $\Pi_{3r}^d$ . Obviously Theorem 4.1 gives the synchronization time of  $3r$  time units for any  $(G_r, x_g, d)$  in  $\Pi_{3r}^d$ . Thus,  $M_{3r}^d$  is a minimum time solution for  $\Pi_{3r}^d$ .

Let  $(G_r, x_g, d)$  be a member of  $\Pi^d - \Pi_s^d$  which has three radial cells  $x_1, x_2$ , and  $x_3$  such that  $x_1$  and  $x_2$  are adjacent

each other and  $\text{dist}_{G_r}(x_1, x_3) = \text{dist}_G(x_2, x_3) = 2r$ .

We denote the set of these  $d$ -connected graph structures by  $\Pi_{3r+1}^d$ . Theorem 1.1 shows that

$$t_{\min}(G_r, x_g, d) \geq L(G_r, x_g, d) + 1 = 3r + 1$$

for any  $(G_r, x_g, d)$  in  $\Pi_{3r+1}^d$ . Theorem 4.1 gives the synchronization time of  $3r+1$  time units for any  $(G_r, x_g, d)$  in  $\Pi_{3r+1}^d$ . Thus,  $M_{3r}^d$  is a minimum time solution for  $\Pi_{3r+1}^d$ .

These results give Theorem 4.2.

**Theorem 4.2** Let  $\Pi_m^d$  be  $\Pi_{3r}^d \cup \Pi_{3r+1}^d \cup \{(G, x_g, d) \mid |G| = 1\}$ .  $M_{3r}^d$  gives the minimum synchronization time for any  $(G_r, x_g, d)$  in  $\Pi_m^d$ .

## Conclusion

We have examined solutions of the firing-squad synchronization problem for some classes of  $d$ -graph structures and graph structures.

In the first part, we have given solutions for the classes of circuit structures, quasi-circuit structures, and some other digraph structures.

In the second part, we have given two solutions for the class of connected graph structures whose synchronization time for  $(G_r, x_g, d)$  are respectively  $4r$  and  $3r+1$  time units where  $G_r$  is a graph with the radius  $r$ .

In the final part, we have given a improved solution for  $d$ -connected graph structures whose synchronization time for  $(G_r, x_g, d)$  is  $3r$  or  $3r+1$  time units depending upon the property of radial cells. Moreover, we have shown that our solution give

the minimum synchronization time for a subclass of connected graph structures.

#### Acknowledgement

The authors would like to thank Professors M. Kimura, M. Nasu, and A. Maruoka, and Mr. S. Okawa in Tohoku University for their useful discussions. Also they are grateful to Professor K. Kobayashi in Tokyo Institute of Technology for his helpful suggestions by which the  $4r$  solution is improved to the  $3r+1$  and the  $3r$  solutions.

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(Received March 14, 1977)

Fig. 1 The scheme of the solution  $M_c$  for  $(C_n, x_0, 1)$ .

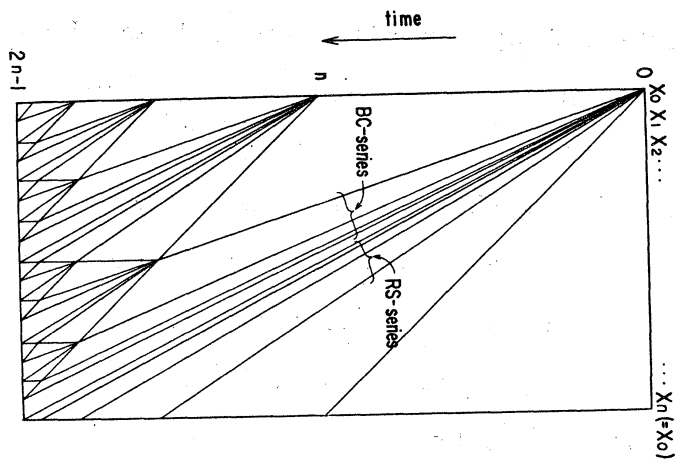


Fig. 2 The solution  $M_c$  for  $(C_{13}, x_0, 1)$ .

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	P <sub>00</sub>												
1	P <sub>1</sub> A <sub>00</sub>												
2	P <sub>0</sub> B <sub>2</sub> <sup>2</sup> A <sub>01</sub>												
3	P <sub>1</sub> B <sub>0</sub> L <sup>1</sup> A <sub>01</sub>												
4	P <sub>0</sub> C <sub>1</sub> <sup>2</sup> T A <sub>01</sub>												
5	P <sub>1</sub> K C <sub>2</sub> S <sub>0</sub> A <sub>00</sub>												
6	P <sub>0</sub> C <sub>0</sub> B <sub>2</sub> <sup>2</sup> S <sub>1</sub> T A <sub>01</sub>												
7	P <sub>1</sub> K B <sub>3</sub> S <sub>2</sub> <sup>1</sup> T A <sub>00</sub>												
8	P <sub>0</sub> K B <sub>2</sub> B <sub>2</sub> <sup>2</sup> R <sub>2</sub> T A <sub>01</sub>												
9	P <sub>1</sub> K B <sub>0</sub> B <sub>3</sub> L <sup>1</sup> R <sub>1</sub> T A <sub>00</sub>												
10	P <sub>0</sub> K C <sub>1</sub> B <sub>2</sub> <sup>2</sup> R <sub>2</sub> T A <sub>01</sub>												
11	P <sub>1</sub> K C <sub>2</sub> B <sub>0</sub> L <sup>1</sup> S <sub>0</sub> T A <sub>00</sub>												
12	P <sub>0</sub> K C <sub>0</sub> C <sub>1</sub> <sup>2</sup> T S <sub>1</sub> T A <sub>01</sub>												
13	P <sub>1</sub> K C <sub>1</sub> C <sub>2</sub> S <sub>0</sub> S <sub>2</sub> T												
14	P <sub>0</sub> A <sub>00</sub> K C <sub>2</sub> C <sub>3</sub> B <sub>2</sub> <sup>2</sup> S <sub>1</sub> R <sub>2</sub> T												
15	P <sub>1</sub> B <sub>1</sub> <sup>1</sup> A <sub>11</sub> K B <sub>0</sub> C <sub>23</sub> S <sub>2</sub> <sup>1</sup> R <sub>1</sub> T												
16	P <sub>0</sub> B <sub>2</sub> L <sup>1</sup> A <sub>00</sub> K C <sub>1</sub> C <sub>3</sub> B <sub>2</sub> <sup>2</sup> S <sub>1</sub> R <sub>2</sub> T												
17	P <sub>1</sub> B <sub>0</sub> L <sup>2</sup> T A <sub>11</sub> K C <sub>2</sub> K C <sub>23</sub> S <sub>2</sub> <sup>1</sup> S <sub>0</sub>												
18	P <sub>0</sub> C <sub>1</sub> S <sub>0</sub> A <sub>00</sub> C <sub>0</sub> C <sub>0</sub> B <sub>2</sub> <sup>2</sup> S <sub>1</sub> T S <sub>1</sub>												
19	P <sub>1</sub> K C <sub>2</sub> B <sub>1</sub> <sup>1</sup> S <sub>1</sub> T P <sub>20</sub> B <sub>3</sub> S <sub>2</sub> <sup>1</sup> T S <sub>2</sub>												
20	P <sub>00</sub> C <sub>0</sub> B <sub>2</sub> S <sub>2</sub> <sup>1</sup> T P <sub>2</sub> P <sub>00</sub> B <sub>2</sub> B <sub>2</sub> <sup>2</sup> R <sub>2</sub>												
21	P <sub>1</sub> A <sub>00</sub> B <sub>3</sub> L <sup>2</sup> R <sub>2</sub> P <sub>2</sub> P <sub>1</sub> A <sub>00</sub> B <sub>3</sub> B <sub>3</sub> L <sup>1</sup> R <sub>1</sub>												
22	P <sub>0</sub> B <sub>2</sub> <sup>2</sup> A <sub>01</sub> B <sub>2</sub> B <sub>2</sub> L <sup>1</sup> P <sub>21</sub> P <sub>0</sub> B <sub>2</sub> <sup>2</sup> A <sub>01</sub> B <sub>2</sub> B <sub>2</sub> <sup>2</sup> R <sub>2</sub>												
23	P <sub>1</sub> B <sub>0</sub> L <sup>1</sup> P <sub>11</sub> B <sub>3</sub> L <sup>2</sup> P <sub>2</sub> P <sub>11</sub> B <sub>0</sub> L <sup>1</sup> P <sub>11</sub> B <sub>3</sub> L <sup>1</sup>												
24	P <sub>0</sub> P <sub>22</sub> C <sub>1</sub> <sup>2</sup> P <sub>0</sub> P <sub>22</sub> B <sub>2</sub> P <sub>22</sub> P <sub>0</sub> P <sub>22</sub> C <sub>1</sub> <sup>2</sup> P <sub>0</sub> P <sub>22</sub> B <sub>2</sub>												
25	F F F F F F F F F F F F F F F F F F												

$n=13$ ,  $S_0 = P_{00}$ ,  $S_1 = F$   
 $S_q$  is denoted by the blank.

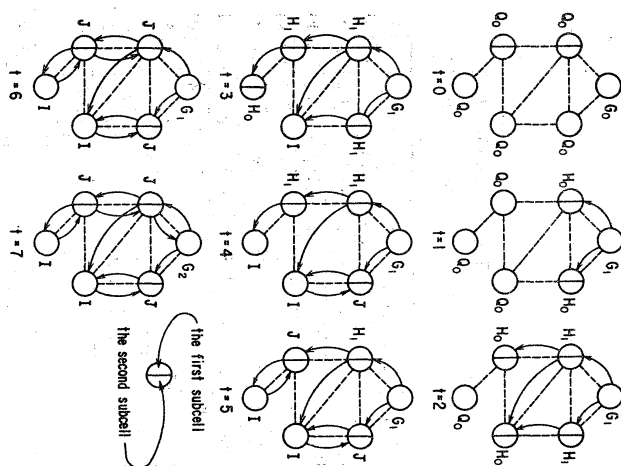


Fig. 3 The illustration of the solution  $M_{4T}^d$  for  $(G_3, x_g, 3)$ .

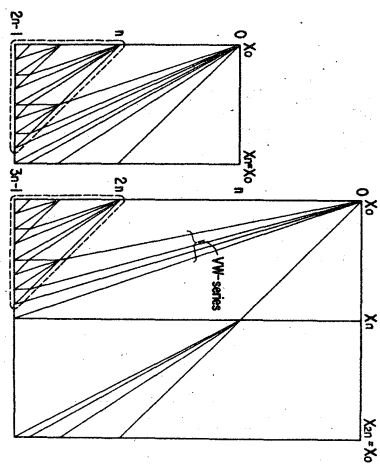


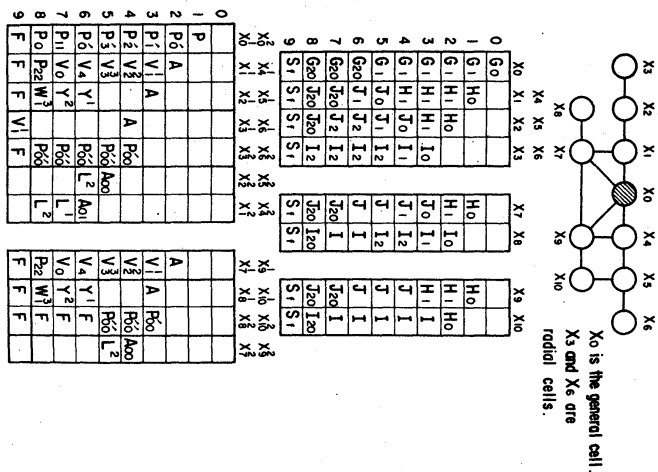
Fig. 4 The scheme of the solution  $M_n$  for  $(C_{2n}, x_0, x_n, 1)$ .

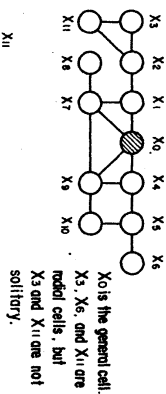
Fig. 5 The solution  $M_h$  for  
( $C_{2,0}$ ,  $x_0$ ,  $x_{10}$ , 1).

	0	1	2	3	4	5	6	7	8	9	10	11
0	P											
1	P <sub>0</sub> A											
2	P <sub>1</sub> V <sub>1</sub> A											
3	P <sub>2</sub> V <sub>2</sub> A											
4	P <sub>3</sub> V <sub>3</sub> A											
5	P <sub>0</sub> V <sub>4</sub> Y <sub>1</sub>											
6	P <sub>1</sub> V <sub>0</sub> Y <sub>2</sub>											
7	P <sub>2</sub> W <sub>3</sub>											
8	P <sub>3</sub> W <sub>2</sub> V <sub>1</sub>											
9	P <sub>0</sub> W <sub>3</sub> V <sub>2</sub>											
10	P <sub>1</sub> K <sub>1</sub> W <sub>4</sub> V <sub>3</sub>											
11	P <sub>2</sub> W <sub>0</sub> V <sub>4</sub> Y <sub>1</sub>											
12	P <sub>0</sub>											
13	P <sub>1</sub> A <sub>00</sub>											
14	P <sub>0</sub> B <sub>2</sub> A <sub>01</sub> V <sub>3</sub> V <sub>4</sub> Y <sub>1</sub> P <sub>00</sub>											
15	P <sub>11</sub> B <sub>0</sub> L <sub>1</sub> P <sub>11</sub> V <sub>5</sub> Y <sub>2</sub> P <sub>00</sub>											
16	P <sub>0</sub> P <sub>22</sub> C <sub>1</sub> <sup>2</sup> P <sub>0</sub> P <sub>22</sub> V <sub>3</sub> P <sub>00</sub>											
17	F	F	F	F	F	F	F	F	F	F	F	F

$n=6$ ,  $S = P_0$ ,  $S_1 = F$   
 $S_q$  is denoted by the blank.

Fig. 6 The solution  $M_{gr}^d$  for ( $G_4$ ,  $x_g$ , 3)  
whose all radial cells are solitary.





	X0	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11
0	G0											
1	G1	H0										
2	G1	H1	H0									
3	G1	H1	H1	H0								
4	G1	H1	H1	I1								
5	G1	H1	J1	I1								
6	G1	J1	J1	I1								
7	G2	J1	J1	I1								
8	G2	J1	J1	I1								
9	G2	J1	J1	I1								
10	S1	S1	S1	S1								

The diagrams for subcells are the same as in Fig. 21

Fig. 7 The solution  $M_{3r}^d$  for  $(G_4, x_g, 3)$  which has non-solitary radial cells.